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BRITISH COLUMBIA

PRINCIPLES OF MATH 12

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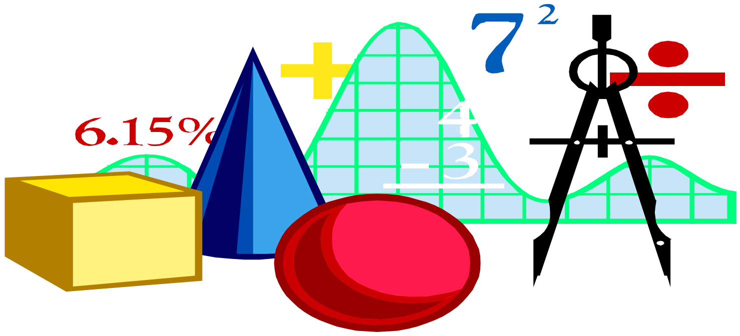
STUDENT
GUIDE AND WORK BOOK



Key to Student Success
with the Principles of Math 12 Curriculum

One of a series of publications by Raven Research Associates for Secondary and Elementary Mathematics

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5.1 Basic Trigonometric Identities

- An equation that is satisfied for all values of the variable for which both sides of the equation are defined is called an **identity**. For example $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is an identity provided $\cos\theta \neq 0$
- There are eight basic trigonometric identities and their family members.

Reciprocal Identities

* $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ $\quad\cos\theta \quad \dfrac{1}{\sec\theta}$ $\quad \tan\theta = \dfrac{1}{\cot\theta}$

- The family members are as follows:

$$\text{cosec} = \frac{1}{\sin\theta} \text{ and } \sin\theta \bullet \operatorname{cosec}\theta = 1$$

$$\sec\theta = \frac{1}{\cos\theta} \text{ and } \cos\theta \bullet \sec\theta = 1$$

$$\cot\theta = \frac{1}{\tan\theta} \text{ and } \tan\theta \bullet \cot\theta = 1$$

- We need only to memorize the first 3(*) because the other family members can be obtained from these with basic algebraic manipulation.

Quotient Identities

* $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

The family members are as follows:

$$\sin\theta = \tan\theta \bullet \cos\theta \text{ and } \cos\theta = \frac{\sin\theta}{\tan\theta}$$

$$\cos\theta = \cot\theta \bullet \sin\theta \text{ and } \sin\theta = \frac{\cos\theta}{\cot\theta}$$

- Again we need only to memorize the first 2 (*)

Pythagorean Identities

* $\sin^2\theta + \cos^2\theta = 1$ $\quad 1 + \tan^2\theta = \sec^2\theta$ $\quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

- The family members follow

$\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta - \tan^2\theta = 1$ and $\tan^2\theta - \sec^2\theta = -1$

$\text{cosec}^2 - \cot^2\theta = 1$ and $\cot^2\theta - \text{cosec}^2 = -1$

- We need only to memorize the first 3(*).

- These identities hold true provided we do not divide by 0 so when $\sec\theta = \frac{1}{\cos\theta}$, the proviso is that $\cos\theta \neq 0$ and when $\tan\theta = \frac{1}{\cot\theta}$, the proviso is that $\cos\theta \neq 0$ and $\sin\theta \neq 0$
- Likewise for the identity $\tan^2\theta + 1 = \sec^2\theta$ the proviso is that $\cos\theta \neq 0$ and for $1 + \cot^2\theta = \mathrm{cosec}^2\theta$ the proviso is $\sin\theta \neq 0$

- A special warning should be noted here for the Pythagorean identity.
    - For example, given 9 + 16 = 25, one might be tempted to take the square root of both sides, which is legal $\sqrt{9+16} = \sqrt{25}$ but $\sqrt{9+16} \neq 3 + 4$ i.e. $3 + 4 \neq 5$
- Likewise we might be tempted to take the root of both sides of $\sin^2\theta + \cos^2\theta = 1$
    - i.e., $\sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1}$ which is legal. but $\sqrt{\sin^2\theta + \cos^2\theta} \neq \sin\theta + \cos\theta$
    $\therefore \sin\theta + \cos\theta \neq 1$
- There are a few methods to verify whether an identity is true or not.
    1. We can choose a specific value for the variable and check to see if the identity is true (but this method only guarantees that the specific value works).
    2. We can graph both sides of the identity on a graphing calculator and check to see if the graphs are identical.
    3. We can algebraically prove by manipulation that both sides of an identity are the same.

Example 1

Verify that $\sin^2 x + \cos^2 x = 1$ for $x = 45^o$

Solution:

L.H.S. *RHS = 1*

$$\sin^2 x + \cos^2 x = \sin^2 45^o + \cos^2 45^o$$
$$= (\sin 45)(\sin 45) + (\cos 45)(\cos 45)$$
$$= \frac{\sqrt{2}}{2} \bullet \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \bullet \frac{\sqrt{2}}{2} = \frac{2}{4} + \frac{2}{4} = \frac{4}{4} = 1$$

$\therefore$ *L.H.S. = R.H.S.*

Example 2

Verify that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ with a graphing calculator.

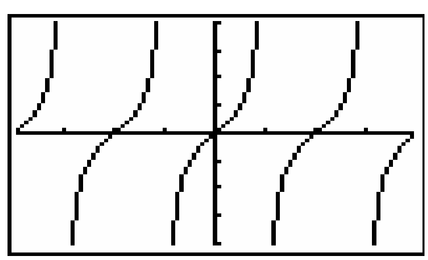
Solution:

Use $y_1 = \tan\theta$

$y_2 = \sin\theta / \cos\theta$

Both graphs are identical, one on top of the other

$\therefore y_1 = y_2$ *and L.H.S. = R.H.S.*



- When proving an identity algebraically there are a few skills that are useful and will be encountered in a lot of identities

Skill 1:
Combining (expressing as a single ratio)

Example:

$$\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Skill 2:
Expressing in terms of the basic sine and cosine functions

Example:

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\frac{\sin\theta}{\cos\theta}}{\sec^2\theta}$$
$$= \frac{2\sin\theta}{\cos\theta} \bullet \cos^2\theta = 2\sin\theta \bullet \cos\theta$$

Skill 3:
Factoring

Example:

$$5\sin^2\theta - 5\cos^2\theta = 5(\sin^2\theta - \cos^2\theta)$$
$$= 5(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)$$

Hints: Proving Trig Identities Algebraically
1. Substitution
2. Change to sinθ's and cosθ's
3. Expand/contract (multiply out/factor)
4. Combine/separate (common denominator/separate terms)
5. Multiply by 1 (conjugate)($\frac{\sin\theta}{\sin\theta}, \frac{1-\cos\theta}{1-\cos\theta}$)

- Generally we set up 2 sides L.H.S. and R.H.S., and work separately and algebraically on the sides, treating them as expressions, not equations, until the sides look identical.

Example of substitution and combining

Example 3: Prove that $\tan\theta + \cot\theta = \sec\theta\csc\theta$

L.H.S.

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \quad \text{substitution}$$
$$\text{com. denominator} = \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \quad \text{combining}$$
$$= \frac{1}{\sin\theta\cos\theta}$$

R.H.S.

$$\sec\theta \,\text{cosec}\theta = \frac{1}{\cos\theta} \bullet \frac{1}{\sin\theta} \quad \text{substitution}$$
$$= \frac{1}{\sin\theta\cos\theta}$$

∴ L.H.S. = R.H.S.

Example of Factoring and Combining

Example 4: Prove that $\sec^2 A - \dfrac{\tan A}{\cos A} = \dfrac{1}{1+\sin A}$

L.H.S.

$\sec^2 A - \dfrac{\tan A}{\cos A} = \dfrac{1}{\cos^2 A} - \dfrac{\tan A}{\cos A}$ substitution

$= \dfrac{1 - \tan A \cos A}{\cos^2 A}$ common denominator

$= \dfrac{1-\sin A}{1-\sin^2 A}$ substitution

$= \dfrac{1-\sin A}{(1+\sin A)(1-\sin A)}$ factoring

$= \dfrac{1}{1+\sin A}$ reducing

R.H.S.

$\dfrac{1}{1+\sin A}$

L.H.S. = R.H.S.

Example of multiplying by conjugate i.e.: 1

Example 5: Prove that $\dfrac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

L.H.S.

$\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{\cos\theta}{1-\sin\theta} \bullet \dfrac{1+\sin\theta}{1+\sin\theta}$ mult. by conj.

$= \dfrac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$ mult. denom.

$= \dfrac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$ substituting

$= \dfrac{1+\sin\theta}{\cos\theta}$ reducing

R.H.S.

$\sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}$ change to sin's/cos's

$= \dfrac{1+\sin\theta}{\cos\theta}$

L.H.S. = R.H.S.

Example of Factoring

Example 6: Prove that $\sin^4 A - \cos^4 A = 2\sin^2 A - 1$

L.H.S.

$\sin^4 A - \cos^4 A = (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)$

factoring

$= 1\,(\sin^2 A - \cos^2 A)$ substitution

$= \sin^2 A - (1 - \sin^2 A)$ substitution

$= \sin^2 A - 1 + \sin^2 A$ distribution

$= 2\sin^2 A - 1$ combining like terms

R.H.S.

$2\sin^2 A - 1$

R.H.S.

Exercises 5.1

1. For the following statements
   (i) verify for the value given.
   (ii) verify using a graphing calculator.
   (iii) prove algebraically.

   a) $\sin^2\theta - \tan\theta + \cos^2\theta = \dfrac{\cos\theta - \sin\theta}{\cos\theta}$, for $\theta = 45^o$

   b) $1 + 2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2$, for $\theta = 30^o$

   c) $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$, for $\theta = 60^o$

2. Prove the following identities.

   a) $\dfrac{\sin\theta\cos\theta}{1-\sin^2\theta} = \tan\theta$

   b) $\dfrac{1-\sin^2\theta}{1-\sin\theta} - 1 = \sin\theta$

   c) $\cos^2\theta(\cot^2\theta + 1) = \cot^2\theta$

   d) $\sin\theta + \cos\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$

   e) $\dfrac{1+\tan\theta}{\sin\theta} = \text{cosec}\,\theta + \sec\theta$

   f) $\dfrac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$

g) $(\sin\theta + \cos\theta)^2 = 2\sin\theta\cos\theta + 1$

h) $\dfrac{1+\tan\theta}{1+\cot\theta} = \tan\theta$

3. Identify the values of the variables in question 2 above for which the identity is undefined in the interval $0 \leq \theta < 2\pi$.

a) $\dfrac{\sin\theta\cos\theta}{1-\sin^2\theta} = \tan\theta$

b) $\dfrac{1-\sin^2\theta}{1-\sin\theta} - 1 = \sin\theta$

c) $\cos^2\theta(\cot^2\theta + 1) = \cot^2\theta$

d) $\sin\theta + \cos\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$

e) $\dfrac{1+\tan\theta}{\sin\theta} = \text{cosec}\theta + \sec\theta$

f) $\dfrac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$

g) $(\sin\theta + \cos\theta)^2 = 2\sin\theta\cos\theta + 1$

h) $\dfrac{1+\tan\theta}{1+\cot\theta} = \tan\theta$